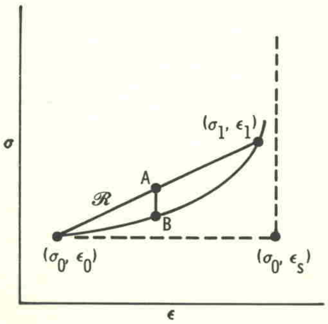

FIG. 2. Typical equilibrium stress-strain curve. Also shown is the Rayleigh line ℛ for a wave taking the material from the state (σ_0, ϵ_0) to the state (σ_1, ϵ_1). The dashed lines represent the behavior of an ideal locking material.

In these coordinates Eq. (1) takes the form

$$\epsilon = -U_\xi, \quad \dot{\epsilon} = -U_{\xi\tau} + VU_{\xi\xi}, \quad u = U_\tau - VU_\xi \quad , \tag{4}$$

and Eq. (2) becomes

$$\sigma_\xi + \rho_0(U_{\tau\tau} - 2VU_{\tau\xi} + V^2U_{\xi\xi}) = 0 \quad . \tag{5}$$

In a steady wave propagating at the velocity V the field variables depend on ξ alone, so all τ derivatives vanish. Equation (4) then takes the form $\epsilon = -dU/d\xi$, $\dot{\epsilon} = Vd^2U/d\xi^2$, and $u = -V\,dU/d\xi$, leading to the important relations

$$\dot{\epsilon} = -V\frac{d\epsilon}{d\xi} \quad \text{and} \quad u = V\epsilon, \tag{6}$$

and Eq. (5) becomes $d(\sigma - \rho_0V^2\epsilon)/d\xi = 0$. This latter equation is readily integrated to give $\sigma - \rho_0V^2\epsilon = \text{const}$. If the material is in a state σ_0, ϵ_0 at some point of the wave (actually we will assume this to be the case as $\xi \to \infty$), the constant can be evaluated and we have

$$\sigma - \sigma_0 = \rho_0V^2(\epsilon - \epsilon_0) \quad . \tag{7}$$

Since the second equation of (6) holds everywhere in the wave, it implies that $u_0 = V\epsilon_0$ and hence that

$$u - u_0 = V(\epsilon - \epsilon_0) \quad , \tag{8}$$

where u_0 is the particle velocity of the material in the state (σ_0, ϵ_0). Equations (7) and (8) together give

$$\rho_0(u - u_0)V - (\sigma - \sigma_0) = 0 \quad , \tag{9a}$$

$$(\epsilon - \epsilon_0)V - (u - u_0) = 0 \quad . \tag{9b}$$

These formulas are of the same form as the Rankine-Hugoniot shock equations. In particular, if u, ϵ, and σ in Eqs. (9) are assigned the values of these quantities behind the wave, then the two pairs of equations are identical when u_0, ϵ_0, and σ_0 refer to values of the unsubscripted variables ahead of the wave. This shows that any steady-wave experiment can be interpreted as a shock experiment if only equilibrium states behind the wave are of interest. As we shall see, the steady-wave analysis gives the complete wave profile. From Eq. (7) we see that the (σ, ϵ) path followed by a particle during the passage of a steady wave is the straight line, called the Rayleigh line ℛ, connecting the initial and final states in the (σ, ϵ) plane, and the wave speed is determined from the slope of this line:

$$\rho_0V^2 = (\sigma_1 - \sigma_0)(\epsilon_1 - \epsilon_0)^{-1} \quad . \tag{10}$$

In order that any existing steady-wave solutions be of practical interest, they must be stable and should also be unique. The heuristic discussion in Sec. I suggests that steady waves will be stable, and demonstrations of this stability in certain cases, as well as approximate solutions to wave evolution problems, have been given by Lighthill[16] and Bland.[19] One of the simpler theories to be discussed in this paper results in the equation $f(U_X)U_{XX} + \nu U_{XtX} = U_{tt}$ which has been studied rather extensively in a recent series of papers[20–22] in which the existence, uniqueness, and stability of steady-wave solutions are discussed.

The properties of steady waves discussed above, in addition to their simplicity, suggest their use in the experimental determination of constitutive equations. Unfortunately, as is apparent from the fact that each member of a broad class of disturbances evolves to the same steady wave, all information bearing on the evolutionary process is lost and one cannot expect steady-wave measurements alone to determine a constitutive equation uniquely. For this reason we must select a general class of constitutive equations as a starting point, and then demonstrate its applicability to the problem at hand. This is done in Secs. III–V.

III. CONSTITUTIVE EQUATIONS

As a starting point for the selection of a constitutive equation we note that experiments conducted on a variety of porous materials show that, for any given material, the states achieved as a result of dynamic compaction lie on a single, unique stress-strain curve $\sigma = \sigma_E(\epsilon)$ that is independent of the rate at which the compaction occurred. This observation suggests our first basic constitutive assumption: When the strain rate becomes zero at the end of a compaction process, the existing stress is a function of the strain, $\sigma = \sigma_E(\epsilon)$. We call this functional relationship an equilibrium stress-strain curve. These curves have been measured for a variety of materials, and several mathematical representations for them have been advanced. The most recent and complete theory of these curves known to us is that of Herrmann.[4]

The experiments that generate the equilibrium curves also show that, while a theory in which the stress is assumed to be a function solely of strain can predict the result of a compaction process correctly, it fails to provide an adequate description of the process itself. The fact that waves induced by planar impact do not propagate as centered simple waves, velocity discontinuities, or combinations thereof is the most typical indication of this failure. In order to obtain a theory capable of describing the

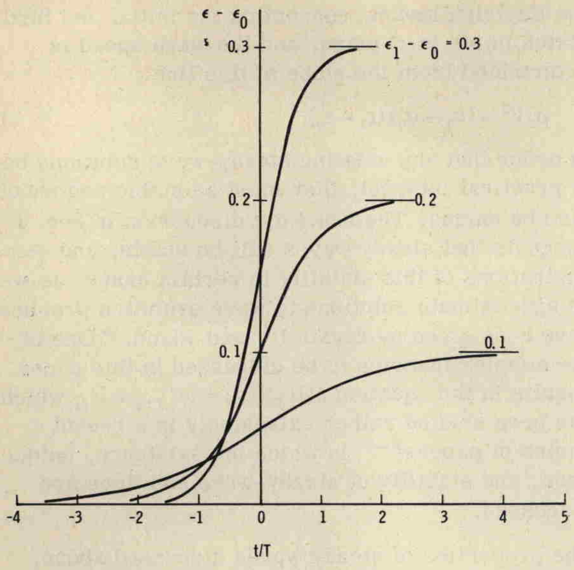

FIG. 3. Steady waveforms of various amplitudes in a material collapsing according to Eq. (20). We have taken $\beta^2 = 10$.

compaction process we generalize the rate-independent theory associated with the constitutive equation $\sigma = \sigma_E(\epsilon)$ by the inclusion of an additional contribution to the stress that is dependent on the rate of straining. This generalization is based on the discussion in the report of Johnson[11] and the paper of Butcher,[12] but focuses on the exploration of collapse phenomena while ignoring some of the range of effects covered in these articles.

Both Johnson and Butcher found it convenient to separate their thinking about material response into two parts dealing with the configuration of the material when loaded but in equilibrium (i.e., when the strain rate is zero) and the strain rate during collapse under applied load, respectively. While this has led to an unusual representation for a constitutive equation that is actually quite conventional, we have found the breakdown to be of great practical value and have continued to use it. The configuration of a body in equilibrium is, of course, obtained from the equilibrium stress-strain curve. The strain rate during a compaction process must be obtained from the complete rate-dependent constitutive equation. Since this equation is framed in a form inverted for strain rate, it is described as a "collapse rule". In the following we will show that the theory obtained in this way can be expressed in the conventional[23] form $\sigma = \sigma_E(\epsilon) + \psi(\epsilon, \dot{\epsilon})$, where $\psi(\epsilon, 0) = 0$.

The specific problem motivating this study lies in the calculations of Butcher which, while including a number of effects not included in this work and employing a very precise representation of the equilibrium stress-strain curve, fail to provide an adequate description of the observed steady-wave profiles in the material studied. The shortcomings seem traceable to the use of an oversimplified collapse rule [Eq. (1.19) of Ref. 12]. In this paper we generalize the linear collapse rule employed by Butcher in a way that seems plausible, fits conveniently into the conventional framework for continuum mechanics, and enables it to accommodate all steady-wave observations exactly.

A. Equilibrium Stress-Strain Curves

As noted previously, the determination of equilibrium stress-strain curves has been the object of many investigations over the past decade. The present work is built on this foundation and the equilibrium stress-strain curves called for in this paper are just those that have been determined before. Since these curves have fairly elaborate mathematical representations [often compounded by their expression in the form $\epsilon = f(\sigma)$], or exist only in graphical or tabular form, calculations using them are done by numerical means.

In all cases we have assumed that the equilibrium response of the material is described by a stress-strain curve that is concave toward the stress axis. For materials exhibiting a yield behavior, this requirement will be met only if the analysis is restricted to the range of states above some stress $\sigma_0 > 0$. Compaction waves propagating in a material having a yield point are unstable and separate into a low-amplitude precursor followed by the slower-propagating main compaction wave. When we take the stress σ_0 to be the precursor amplitude, then the present analysis is applicable to the description of the main compaction wave.

In order to accomplish the parameter studies of Sec. IV, it is convenient to have at hand a mathematical representation of the equilibrium stress-strain curve that, in addition to providing a reasonable approximation to real-material behavior at low strains, (a) is simple enough to allow analytical calculation of steady-wave forms for a variety of collapse rules, (b) involves a single dimensionless parameter that measures the departure from linearity, and (c) does not contribute to asymmetry of calculated steady waveforms. The function

$$\sigma = \sigma_0 + \rho_0 c_0^2 (\epsilon - \epsilon_0) [1 + \beta_2 (\epsilon - \epsilon_0)] \quad (11)$$

fulfills these requirements; it approximates the observed behavior at low strains where dispersion effects are important and involves the parameter β characterizing the nonlinearity. It also eliminates the equilibrium behavior as a contributor to asymmetry of waveforms, but a discussion of what this means and a demonstration that it is accomplished must be postpone until Sec. IV. The extreme example of nonlinear equilibrium response is provided by the locking model represented by the dashed lines on Fig. 2 and discussed at the end of Sec. IV. The specific forms of the stress-strain curve given by Eq. (11) or by the locking model are not, of course, central features of the theory; they